Week 04: Linearization and State Models

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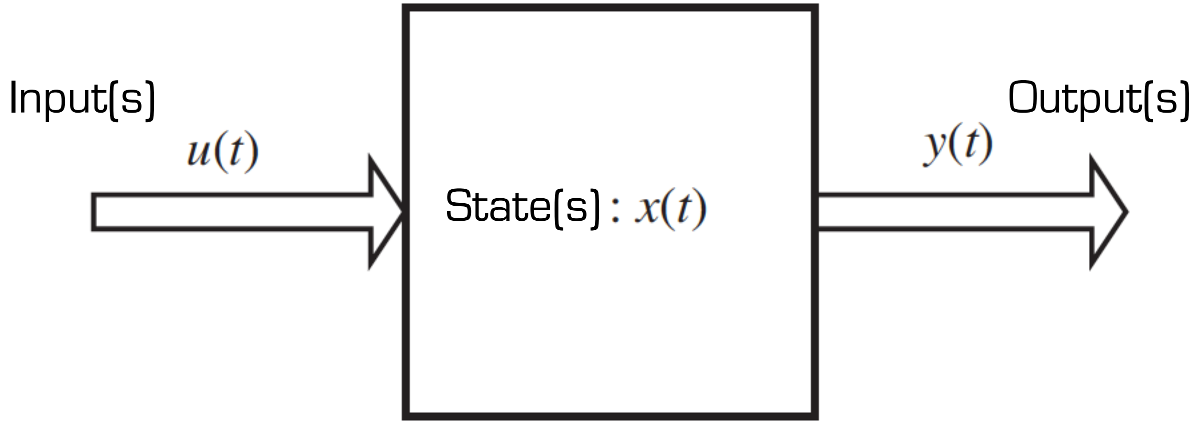
Lecture Overview

- State-space Approach to Modeling Dynamical Systems
- Linearization of Non-linear Systems

State-space Approach to Modeling Dynamical Systems



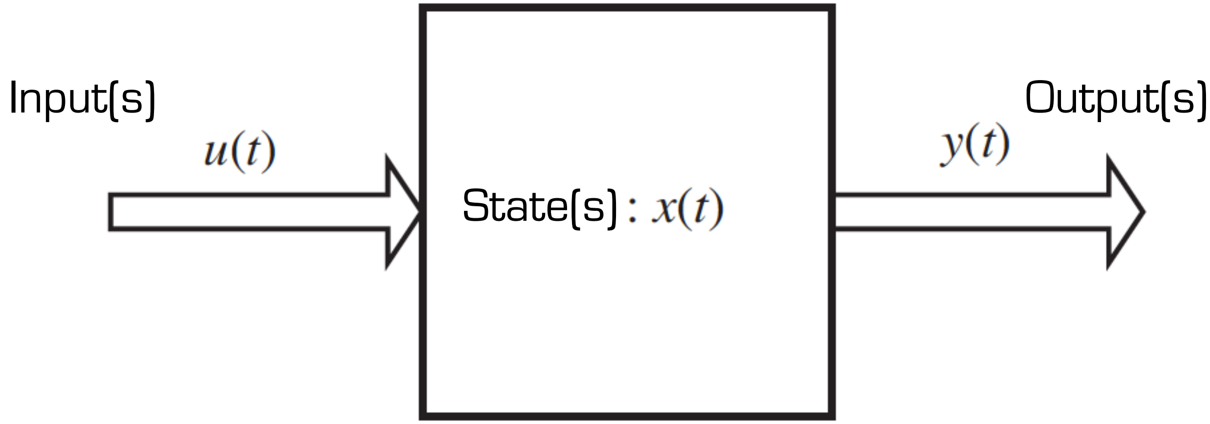
General Form

$$\dot{x}(t) = f[x(t), u(t), t]$$

$$y(t) = g[x(t), u(t), t]$$

$$x(t_0) = x_0$$

State-space Approach to Modeling Dynamical Systems



Linear and Time-Invariant System

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$$x(0) = x_0$$

States

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Solution of the State Equation

- **Homogenous State Equation**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$\mathbf{x} = n$-vector

$\mathbf{A} = n \times n$ constant matrix

$$\mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots + \frac{1}{k!}\mathbf{A}^kt^k + \cdots\right)\mathbf{x}(0)$$

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)}$$

Exponentials of Matrices

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} \qquad e^{\mathbf{A}(t+s)} = e^{\mathbf{A}t}e^{\mathbf{A}s}$$

$$e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t}e^{\mathbf{B}t}, \qquad \text{if } \mathbf{AB} = \mathbf{BA}$$

Solution of the State Equation

- **Non-Homogenous State Equation**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\begin{aligned} \mathbf{x} &= n\text{-vector} \\ \mathbf{u} &= r\text{-vector} \\ \mathbf{A} &= n \times n \text{ constant matrix} \\ \mathbf{B} &= n \times r \text{ constant matrix} \end{aligned}$$

$$e^{-\mathbf{A}t}[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)] = \frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau}$$

Another perspective

- From an n^{th} order differential equation to n 1^{st} order differential equations

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_0 y(t) = bu(t)$$

$$y^{(i)}(0),\ i = 0, 1, \ldots, n-1$$

State Variables

$$\begin{aligned}
x_1(t) &:= y(t) & \dot{x}_1(t) &= x_2(t) \\
x_2(t) &:= y^{(1)}(t) & \dot{x}_2(t) &= x_3(t) \\
&\vdots & &\vdots \\
x_{n-1}(t) &:= y^{(n-2)}(t) & \dot{x}_{n-1}(t) &= x_n(t) \\
x_n(t) &:= y^{(n-1)}(t) & \dot{x}_n(t) &= -a_{n-1}x_n(t) - a_{n-2}x_{n-1}(t) - \ldots - a_0 x_1(t) + bu(t)
\end{aligned}$$

Another perspective

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}$$

$$x_1(0),\ x_2(0),\ \dots,\ x_n(0)$$

Utility of State-space Representation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
$$x(0) = x_0$$

- **COMPUTATION**
  - Capability of high-speed solution of differential equations
  - The format of state-space representation is specifically developed for computer algorithms and simulation framework

Utility of State-space Representation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
$$x(0) = x_0$$

- **Stability:** eigenvalues of the A matrix

- **Controllability:** the ability of an external input (the vector of control variables) to move the internal state of a system from any initial state to any other final state in a finite time interval
  - The rank of the controllability matrix (A and B matrices)

- **Observability:** one can determine the behavior of the entire system from the system's outputs
  - The rank of the observability matrix (A and C matrices)

Definitions

- The state of a dynamical system is the smallest set of variables such that knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$ completely determines the behavior for any time $t \geq t_0$.

- The state of an LTI dynamical system at time t is independent of the state and input before t_0 (initially at rest).

- Variables that do not represent physical quantities can be chosen as state variables

- **How do we know the minimum number of states?**
  - The minimum number required states equals to the **order of the differential equations** describing the system

Example 1 (Non-linear System)

$$a\ddot{w} + \sin\dot{z} = u_1^{\,2}$$
$$\sqrt{\dot{v}} + \cos\dot{z} = u_2$$
$$\dot{z} + z = \alpha t$$

$$v(t_0) = v_0,\ w(t_0) = w_0$$
$$\dot{w}(t_0) = a_0 \qquad z(t_0) = z_0$$

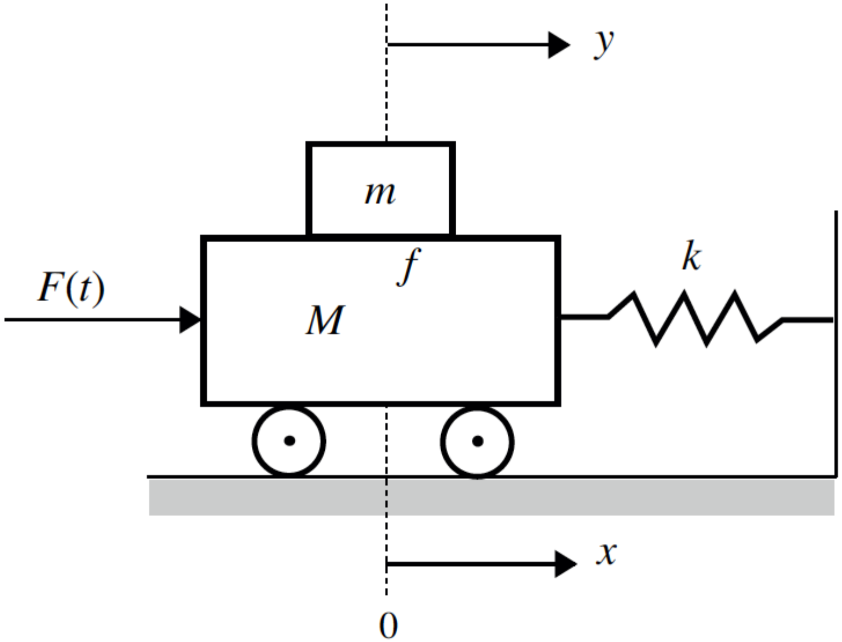
State Variables

$$x_1 := v$$
$$x_2 := w\ ,\ x_3 := \dot{w}$$
$$x_4 := z$$

State Equations

$$\dot{x}_1 = \dot{v} = (u_2 - \cos\dot{z})^2 = [u_2 - \cos(-x_4 + \alpha t)]^2$$
$$\dot{x}_2 = \dot{w} = x_3$$
$$\dot{x}_3 = \ddot{w} = \frac{1}{a}[u_1^2 - \sin(-x_4 + \alpha t)]$$
$$\dot{x}_4 = \dot{z} = -x_4 + \alpha t$$

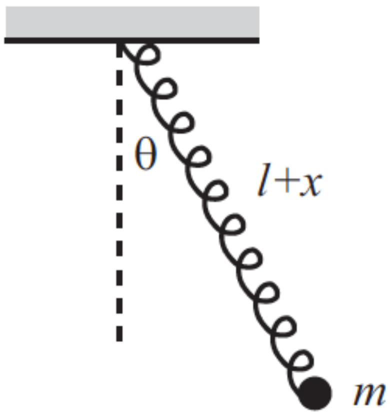
Example 2



$$M\ddot{x} = -kx - f(\dot{x} - \dot{y}) + F$$

$$m\ddot{y} = f(\dot{x} - \dot{y})$$

Example 3: Spring Pendulum



- A pendulum made of a spring with a mass m on the end
- The equilibrium length of the spring is l
- Assume that the motion takes place in a vertical plane
- Find equations of motion for generalized coordinates x and θ

$$T = \frac{1}{2}m\left(\dot{x}^2 + (l+x)^2\dot{\theta}^2\right)$$

$$V = -mg(l+x)\cos\theta + \frac{1}{2}kx^2$$

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + (l+x)^2\dot{\theta}^2\right) + mg(l+x)\cos\theta - \frac{1}{2}kx^2$$

Example 3: Spring Pendulum

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + (l+x)^2\dot{\theta}^2\right) + mg(l+x)\cos\theta - \frac{1}{2}kx^2$$

- Note that there are two generalized coordinates, x and θ.

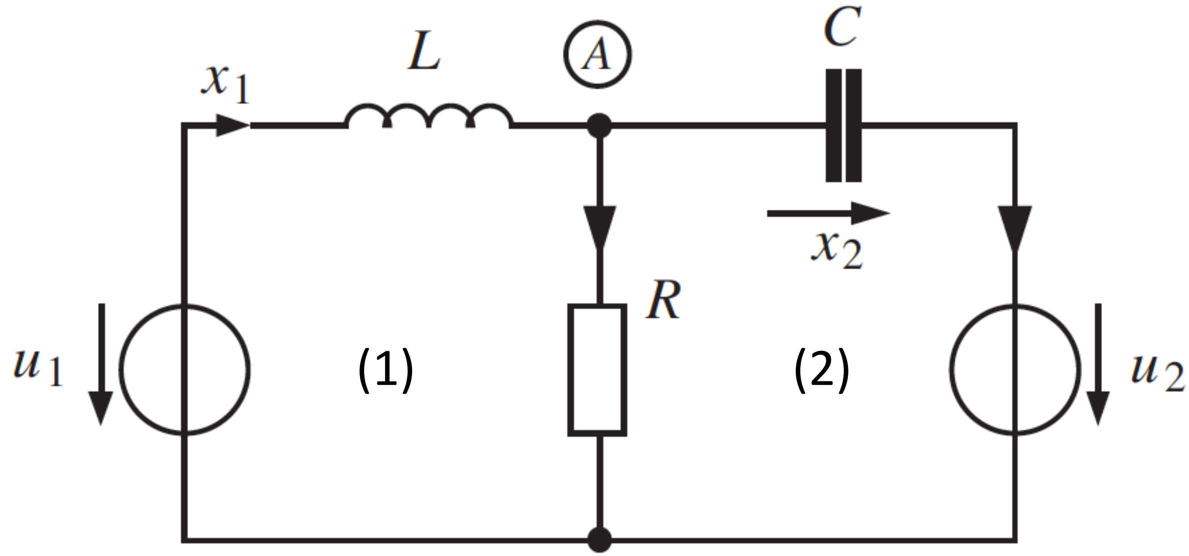
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \quad \rightarrow \quad m\ddot{x} = m(l+x)\dot{\theta}^2 + mg\cos\theta - kx \quad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \quad \rightarrow \quad \frac{d}{dt}\left(m(l+x)^2\dot{\theta}\right) = -mg(l+x)\sin\theta$$

$$m(l+x)^2\ddot{\theta} + 2m(l+x)\dot{x}\dot{\theta} = -mg(l+x)\sin\theta$$

$$m(l+x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta \qquad (2)$$

Example 4



- Inputs: voltages u_1 and u_2

- x_1 : current flowing through inductor
- x_2 : voltage on the capacitor

State-space modeling with input derivatives

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

Define:

$$x_1 = y - \beta_0 u \qquad \beta_0 = b_0$$

$$x_2 = \dot{x}_1 - \beta_1 u \qquad \text{where} \qquad \beta_1 = b_1 - a_1\,\beta_0$$

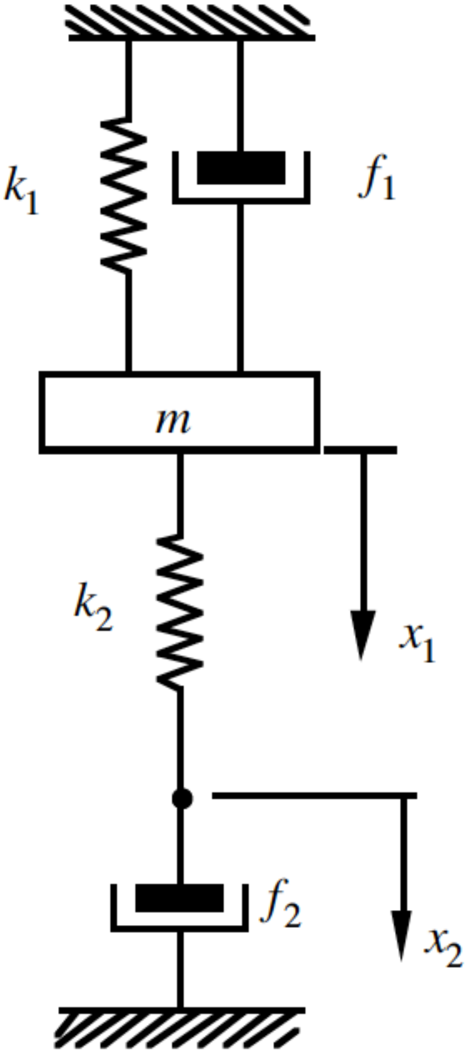
$$y = x_1 + \beta_0 u$$

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + \beta_2 u$$

$$\dot{x}_1 = x_2 + \beta_1 u \qquad \text{where} \qquad \beta_2 = b_2 - a_1\,\beta_1 - a_2\,\beta_0$$

$$y = x_1 + \beta_0 u$$

Example 5



- Input: displacement x_2
- Output: displacement x_1

$$m\ddot{x}_1 = -k_1 x_1 - f_1 \dot{x}_1 - k_2(x_1 - x_2)$$

$$0 = -f_2 \dot{x}_2 + k_2(x_1 - x_2)$$

$$m\ddot{x}_1 = -k_1 x_1 - f_1 \dot{x}_1 - f_2 \dot{x}_2$$

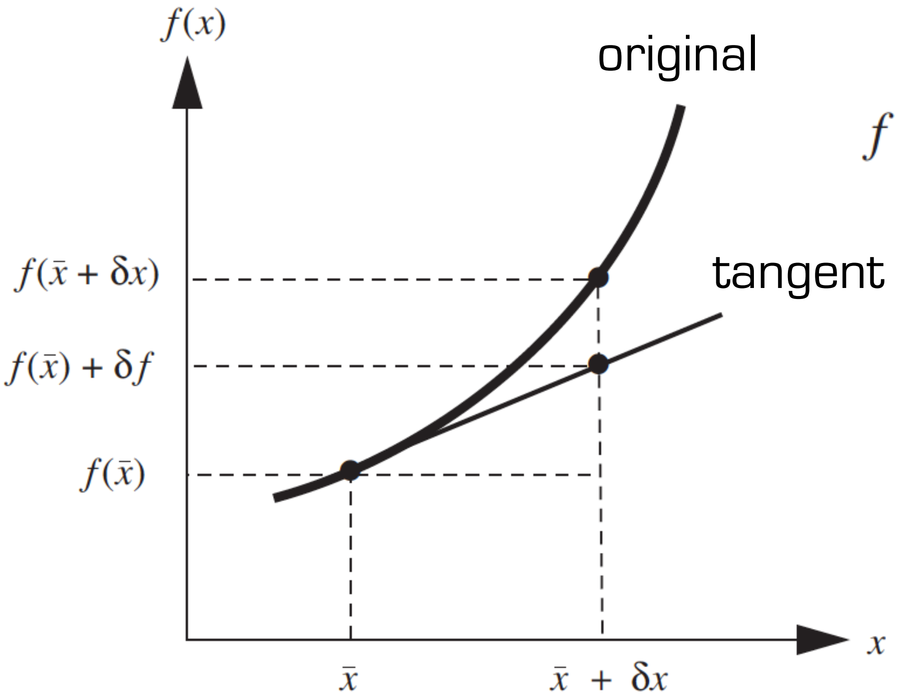
$$\ddot{y} + \frac{f_1}{m}\dot{y} + \frac{k_1}{m}y = -\frac{f_2}{m}\dot{u}$$

Linearization of Nonlinear Systems

- Linear Approximation
  - Approximation of individual nonlinear terms
- **Jacobian Linearization**

Tangent Line Approximation and Taylor's Theorem

- For small deviations δx around a fixed point $\bar{x}$



Taylor Series

$$f(\bar{x}+\delta x) = f(\bar{x}) + \frac{df}{dx}\delta x + \frac{1}{2}\frac{d^2 f}{dx^2}(\delta x)^2 + \ldots$$

Approximation

$$f(\bar{x}+\delta x) \approx f(\bar{x}) + \frac{df}{dx}(\bar{x})\delta x$$

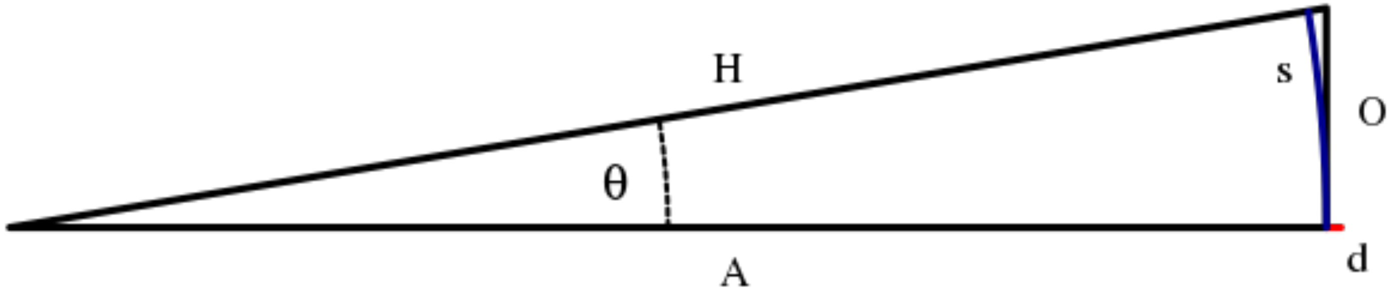
$$\delta f = \frac{df}{dx}(\bar{x})\delta x$$

Small Angle Approximation

- Approximate the values of main trigonometric functions
  - **The angle is small and measured in radians**

$$sin\theta \approx \theta \qquad tan\theta \approx \theta$$

$$cos\theta \approx 1 \qquad sin^2\theta \approx 0$$



$$sin\theta = \frac{O}{H} \approx \frac{O}{A} \approx \frac{s}{A} = \frac{A\theta}{A} = \theta$$

Small Angle Approximation

- Approximate the values of main trigonometric functions
  - **The angle is small and measured in radians**

$$sin\theta \approx \theta \qquad tan\theta \approx \theta$$

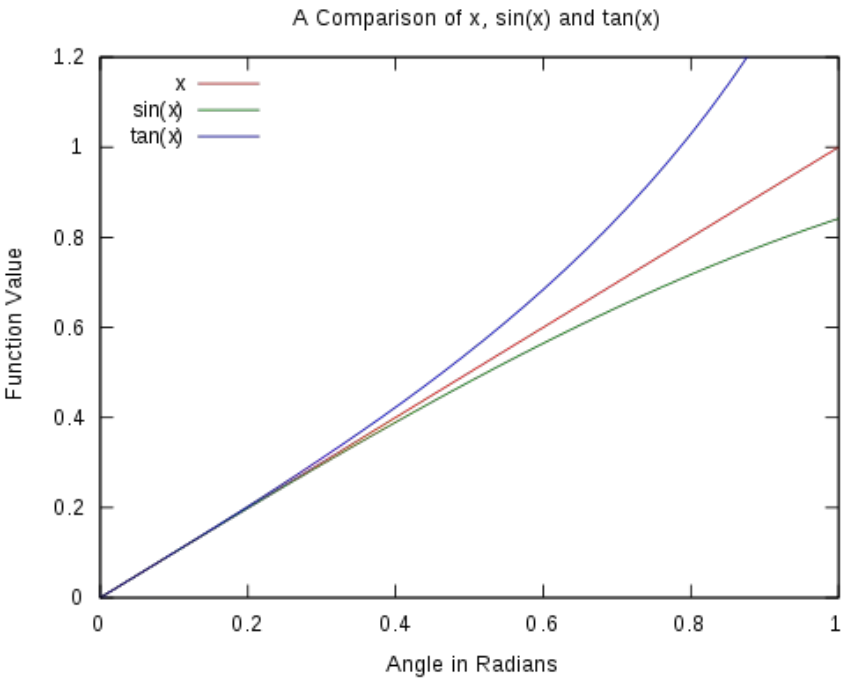
$$cos\theta \approx 1 \qquad sin^2\theta \approx 0$$

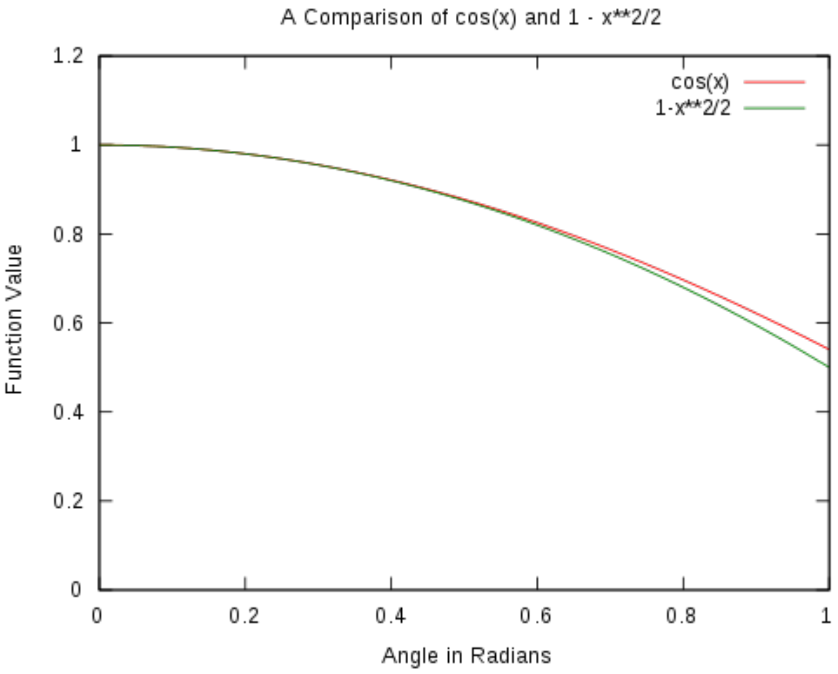
Taylor series expansion about 0

$$sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1} = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}$$

Small Angle Approximation

- Approximate the values of main trigonometric functions
  - **The angle is small and measured in radians**





Equilibrium Point

$$\dot{x}(t) = f[x(t), u(t)]$$

$$y(t) = g[x(t), u(t)]$$

$$x(0) = x_0$$

- At the equilibrium point $(\bar{u}, \bar{x}, \bar{y})$, the derivatives will go to zero.

$$0 = f[\bar{x}, \bar{u}]$$

$$\bar{y} = g[\bar{x}, \bar{u}]$$

Taylor Series Approximation

$$\dot{x} = f[\bar{x}, \bar{u}] + \left.\frac{\partial f}{\partial x}\right|_{\bar{u}, \bar{x}} (x - \bar{x}) + \left.\frac{\partial f}{\partial u}\right|_{\bar{u}, \bar{x}} (u - \bar{u})$$

$$y = g[\bar{x}, \bar{u}] + \left.\frac{\partial g}{\partial x}\right|_{\bar{u}, \bar{x}} (x - \bar{x}) + \left.\frac{\partial g}{\partial u}\right|_{\bar{u}, \bar{x}} (u - \bar{u})$$

Approximation in Matrix Form

- Introduce variables for small variations:

$$\delta x(t) := x(t) - \bar{x} \qquad \delta u(t) := u(t) - \bar{u} \qquad \delta y(t) := y(t) - \bar{y}$$

- Note that $\delta\dot{x} = \dot{x}$

$$\delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{\bar{u},\,\bar{x}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{\bar{u},\,\bar{x}} \delta u \qquad \delta y = \left.\frac{\partial g}{\partial x}\right|_{\bar{u},\,\bar{x}} \delta x + \left.\frac{\partial g}{\partial u}\right|_{\bar{u},\,\bar{x}} \delta u$$

Linearized Version of State Model

$$\delta\dot{x} = A\,\delta x + B\,\delta u$$

$$\delta y = C\,\delta x + D\,\delta u$$

$$\delta x(0) = x_0 - \bar{x}$$

$$A := \left.\frac{\partial f}{\partial x}\right|_{\bar{u},\,\bar{x}} \qquad C := \left.\frac{\partial g}{\partial x}\right|_{\bar{u},\,\bar{x}}$$

$$B := \left.\frac{\partial f}{\partial u}\right|_{\bar{u},\,\bar{x}} \qquad D := \left.\frac{\partial g}{\partial u}\right|_{\bar{u},\,\bar{x}}$$

Jacobian Linearization

$$A := \left.\frac{\partial f}{\partial x}\right|_{\bar{u},\,\bar{x}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\bar{u},\,\bar{x}}$$

$(n \times n)$

$$B := \left.\frac{\partial f}{\partial u}\right|_{\bar{u},\,\bar{x}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_p} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_p} \end{bmatrix}\right|_{\bar{u},\,\bar{x}}$$

$(n \times p)$

$$C := \left.\frac{\partial g}{\partial x}\right|_{\bar{u},\,\bar{x}} = \left.\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_q}{\partial x_1} & \frac{\partial g_q}{\partial x_2} & \cdots & \frac{\partial g_q}{\partial x_n} \end{bmatrix}\right|_{\bar{u},\,\bar{x}}$$

$(q \times n)$

$$D := \left.\frac{\partial g}{\partial u}\right|_{\bar{u},\,\bar{x}} = \left.\begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_p} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & \cdots & \frac{\partial g_2}{\partial u_p} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_q}{\partial u_1} & \frac{\partial g_q}{\partial u_2} & \cdots & \frac{\partial g_q}{\partial u_p} \end{bmatrix}\right|_{\bar{u},\,\bar{x}}$$

$(q \times p)$

Example 1

$$\dot{x} = -2x + 0{,}5(x+1)u$$

Linearize the system around the constant input at $\bar{u} = 2$

- Equilibrium Point

$$0 = -2\bar{x} + 0{,}5(\bar{x}+1)\bar{u} \qquad \bar{u} = 2 \qquad \bar{x} = 1$$

- Jacobian Linearization

$$f(x,u) = -2x + 0{,}5xu + 0{,}5u$$

$$A = \left.\frac{\partial(-2x + 0{,}5xu + 0{,}5u)}{\partial x}\right|_{\bar{u},\,\bar{x}} = -2 + 0{,}5\bar{u} = -1$$

$$B = \left.\frac{\partial(-2x + 0{,}5xu + 0{,}5u)}{\partial u}\right|_{\bar{u},\,\bar{x}} = 0{,}5\bar{x} + 0{,}5 = 1$$

Example 1

Linearized Version of State Model

$$\delta\dot{x} = A\,\delta x + B\,\delta u$$

$$\delta\dot{x} = -\,\delta x + \delta u$$

Example 2

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{dx_2(t)}{dt} = -\frac{k}{m}x_2(t) - \frac{g}{\ell}\sin x_1(t)$$

- Equilibrium Point

$$x_1 = 0, \quad x_2 = 0$$

- Small angle approximation (Taylor Series for x_1=0)

$$\sin\theta = \sin 0 + \cos 0\,(\theta - 0) - \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \cdots$$

$$\boxed{\sin\theta \approx \theta}$$

Example 2

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{dx_2(t)}{dt} = -\frac{k}{m}x_2(t) - \frac{g}{\ell}\sin x_1(t)$$

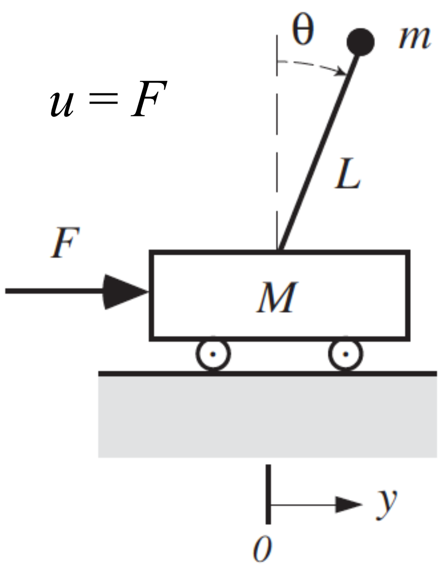
- Equilibrium Point

$$x_1 = 0, \quad x_2 = 0$$

- Jacobian Linearization

$$A = \begin{bmatrix} \frac{\partial f_1(x_1,x_2)}{\partial x_1} & \frac{\partial f_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial f_2(x_1,x_2)}{\partial x_1} & \frac{\partial f_2(x_1,x_2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & \frac{-k}{m} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & \frac{-k}{m} \end{bmatrix}$$

Example 3



$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 \ell \sin\theta - g\sin\theta\cos\theta\right)$$

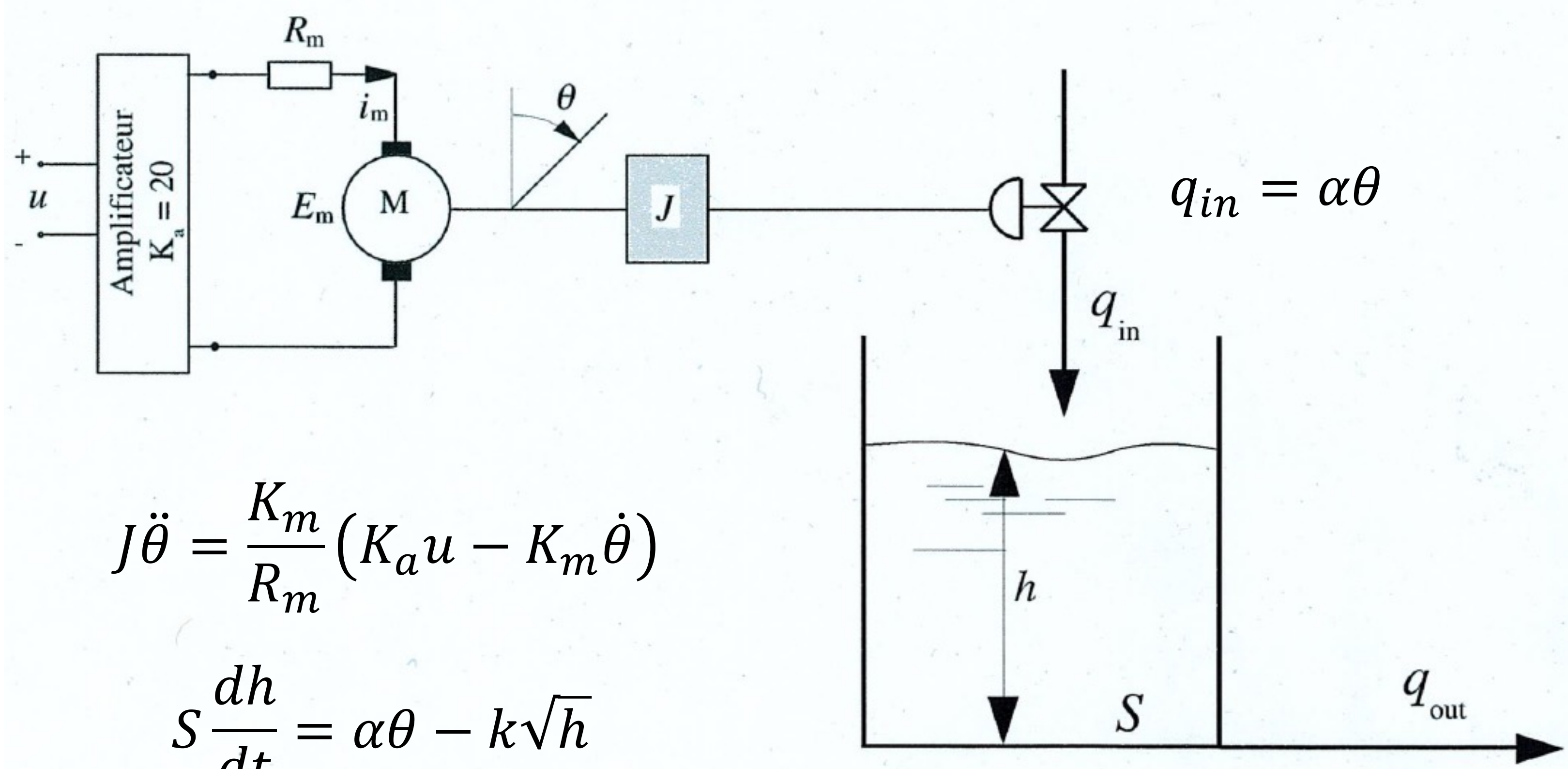
$$\ddot{\theta} = \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2\ell\cos\theta\sin\theta + \frac{M+m}{m}g\sin\theta\right)$$

- Equilibrium Point $\theta = 0, \dot{\theta} = 0, \dot{y} = 0$
- Small angle approximation $\sin\theta \approx \theta$ $\cos\theta \approx 1$

$$\ddot{y} = -\frac{m}{M}g\theta + \frac{1}{M}u$$

$$\ddot{\theta} = \frac{M+m}{M\ell}g\theta - \frac{1}{M\ell}u$$

Example 4



$$q_{in} = \alpha\theta$$

$$J\ddot{\theta} = \frac{K_m}{R_m}\left(K_a u - K_m\dot{\theta}\right)$$

$$S\frac{dh}{dt} = \alpha\theta - k\sqrt{h}$$

$$q_{out} = k\sqrt{h}$$

Linearization: $\sqrt{h} \cong \sqrt{\bar{h}} + \frac{1}{2\sqrt{\bar{h}}}\delta h$

Example 5: Spring Pendulum

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = (l + x_1){x_4}^2 + g\cos x_3 - \frac{k}{m}x_1$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{2x_2x_4}{l + x_1} - \frac{g\sin x_3}{l + x_1}$$